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Construction of the Faithful Irreducible Representation for the Subgroup G Contained in S_7

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Abstract

Every element in the transitive subgroup G of order 42 contained in the symmetric group of degree 7 is expressed as an ordered product of powers of the basic permutations $g_2 \equiv (1234567)$ and $g_8 \equiv (1)(243756)$. The only faithful irreducible representation for G is then given explicitly in terms of the six-dimensional matrices $\mathrm{D}(g_2)$ and $\mathrm{D}(g_8)$, obtained here with the aid of a simple computational algorithm.

The transitive subgroup G of order 42 contained in S_7, the symmetric group of degree 7, is of contemporary physical interest in connection with recent work on quantum electrodynamics (Rosen, 1976). Although the characters of irreducible representations for G have been known for many years, a discussion of the group structure and an explicit derivation of the faithful six-dimensional representation has not been reported heretofore in the literature.[1] The purpose of the present communication is to show that every element of G is expressible as an ordered product of powers of the two (noncommuting) basic permutations and to give the only faithful irreducible representation for G explicitly in terms of six-dimensional matrices.

The group G arises if one considers restricted arrangements of seven (objects or) symbols at the vertices of a heptagon. Let a phase factor be associated with each symbol, $x_k \leftrightarrow \exp(2\pi ik/7)$ for $k = 1, \ldots, 7$, and define an *admissible arrangement* to be such that the square of each phase factor associated with a symbol is equal to the product of the phase factors ascribed to the symbols at the two adjacent vertices [equivalently $x_{k'}, x_{k''}, x_{k'''}$ are admissible at three successive vertices if and only if $2k'' = k' + k'''$ (mod 7)]. Then the basic arrangement with x_1 through x_7 set down in arithmetic order at successive vertices of the heptagon is clearly one such admissible arrangement. Moreover, any

[1] This group is, however, the subject of a problem in I. D. Macdonald, *The Theory of Groups* (Oxford University Press, London, 1968) p. 211, where the reader is asked to prove that G is soluble and has Sylow subgroups of order 2 and 3 that do not lie in the same Sylow basis.

arrangement obtained by applying a permutation in G to the basic arrangement is also admissible, while a general permutation in S_7 that is not in G produces an inadmissible arrangement of the seven symbols. If any two of the symbols are set down at two adjacent vertices, then the other five symbols are assigned uniquely to the other five vertices for an admissible arrangement, the $7 \times 6 = 42$ admissible arrangements being in one-to-one correspondence with the elements of G.

Table 1 displays the elements of S_7 that are in the subgroup G with $g_1 \equiv (1)(2)(3)(4)(5)(6)(7)$ the identity operation and the basic permutation of order 7 prescribed as $g_2 \equiv (1234567)$, by definition. The enumeration of the group elements in Table 1 is particularly useful, for detailed calculation

TABLE 1. Elements of G expressed in standard cycle notation with indices k, l in formula (1), where $r = 7k + l + 1$.[a]

r	g_r	k	l	r	g_r	k	l
1	(1)(2)(3)(4)(5)(6)(7)	0	0	22	(1)(27)(36)(45)	3	0
2	(1234567)	0	1	23	(5)(12)(37)(46)	3	1
3	(1357246)	0	2	24	(2)(13)(47)(56)	3	2
4	(1473625)	0	3	25	(6)(14)(23)(57)	3	3
5	(1526374)	0	4	26	(3)(15)(24)(67)	3	4
6	(1642753)	0	5	27	(7)(16)(25)(34)	3	5
7	(1765432)	0	6	28	(4)(17)(26)(35)	3	6
8	(1)(243756)	1	0	29	(1)(253)(467)	4	0
9	(4)(125763)	1	1	30	(3)(126)(475)	4	1
10	(7)(132645)	1	2	31	(5)(134)(276)	4	2
11	(3)(146527)	1	3	32	(7)(142)(356)	4	3
12	(6)(153472)	1	4	33	(2)(157)(364)	4	4
13	(2)(167354)	1	5	34	(4)(165)(237)	4	5
14	(5)(174236)	1	6	35	(6)(173)(245)	4	6
15	(1)(235)(476)	2	0	36	(1)(265734)	5	0
16	(7)(124)(365)	2	1	37	(6)(127435)	5	1
17	(6)(137)(254)	2	2	38	(4)(136752)	5	2
18	(5)(143)(267)	2	3	39	(2)(145376)	5	3
19	(4)(156)(273)	2	4	40	(7)(154623)	5	4
20	(3)(162)(457)	2	5	41	(5)(163247)	5	5
21	(2)(175)(346)	2	6	42	(3)(172564)	5	6

[a] Constituting a conjugate class, the 7 elements with $k = 3$ describe the reflection symmetries of a regular heptagon, while the subgroup composed of the 14 elements with $k = 0, 3$ describes the full symmetry of the regular heptagon. Of all 840 elements in S_7 with the cycle structure $\{1, 6\}$, only the 14 permutations shown in the Table with $k = 1, 5$ are admissible in a proper subgroup of S_7. Likewise, the only 7 permutations with the cycle structure $\{1, 2^3\}$ that admit inclusion in G, where $g_2 = (1234567)$ by definition, are the regular heptagon reflection symmetries with $k = 3$ in the Table.

shows that every permutation in G is expressible as a unique ordered product of powers of g_2 and $g_8 = (1)(243756)$,

$$g_r = g_1(g_2)^l(g_8)^k \tag{1}$$

in which k ($\leqq 5$) is the greatest non-negative integer less than or equal to $(r-1)/7$, and $l \equiv r - 1 - 7k$ ($\leqq 6$). To verify formula (1) with the k, l index assignments in Table 1, the powers and products of g_2 and g_8 factors are simply evaluated by permutation multiplication (e.g., $g_2 g_8 = (4)(125763) \equiv g_9$). The closure of G follows from formula (1) as a consequence of the relations involving the basic permutations,

$$(g_2)^7 = g_1 = (g_8)^6, \qquad g_8 g_2 = (g_2)^3 g_8 \tag{2}$$

with the latter equation yielding $g_8(g_2)^n = (g_2)^{3n} g_8$ for arbitrary integer n by iteration.

Table 2 exhibits the characters of the 7 irreducible representations of G, obtained by application of the general theory (Littlewood, 1940). The only faithful irreducible representation is six-dimensional, with matrices denoted in the following as $\{M_r = \mathbf{D}(g_r)\}$. Using the enumeration of the group elements in Table 1, formula (1) imples that

$$M_r = (M_2)^l(M_8)^k, \qquad r = 7k + l + 1 \tag{3}$$

where $M_1 = \mathbb{1}$ is the six-dimensional identity matrix. The basic matrices M_2 and M_8 in (3) are obtained explicitly by making use of the following computational algorithm.

Let $x_1, \ldots, x_7$ denote 7 symbols that are rearranged by the permutations in G: $x_i \to x_{i_r}$, where $i \to i_r$ under g_r. Define $y_i \equiv x_i - x_{i+1}$ for $i = 1, \ldots, 6$. Then a permutation g_r of the x's induces the transformation

$$y_i \to y_i^{(r)} \equiv x_{i_r} - x_{(i+1)_r} \tag{4}$$

Because the faithful irreducible representation for G is obtainable from the $\mathbf{6} \oplus \mathbf{1}$ reduction of the seven-dimensional permutation matrices in the primary representation of S_7, the matrix elements of M_r appear as the coefficients in the expansion

$$y_k^{(r)} = \sum_{j=1}^{6} y_j (M_r)_{jk} \tag{5}$$

the $\mathbf{6} \oplus \mathbf{1}$ reduction being effected by transformation from the original basis vector $(x_1, x_2, \ldots, x_6, x_7)$ to the new basis vector

$$(y_1, y_2, \ldots, y_6, \sum_{i=1}^{7} x_i)$$

Example: $r = 2$. We have $i_2 = i + 1$ for $i = 1, \ldots, 6$ and $7_2 = 1$. Thus, $y_i^{(2)} = y_{i+1}$ for $i = 1, \ldots, 5$ and

$$y_6^{(2)} = x_7 - x_1 = -\sum_{j=1}^{6} y_j$$

TABLE 2. Characters (Littlewood, 1940) of the irreducible representation of G, where ω is a complex cube root of unity.

Cycles: Conjugate classes	$\{1^7\}$ $(k=0=l)$	$\{7\}$ $(k=0\neq l)$	$\{1,6\}$ $(k=1)$	$\{1,3^2\}$ $(k=2)$	$\{1,2^3\}$ $(k=3)$	$\{1,3^2\}$ $(k=4)$	$\{1,6\}$ $(k=5)$
Order: Number of elements	1	6	7	7	7	7	7
Characters of faithful rep: Trace of M_r's	6	-1	0	0	0	0	0
Characters of unfaithful one-dimensional representations	1	1	1	1	1	1	1
	1	1	ω	ω^2	1	ω	ω^2
	1	1	ω^2	ω	1	ω^2	ω
	1	1	$-\omega$	ω^2	-1	ω	$-\omega^2$
	1	1	$-\omega^2$	ω	-1	ω^2	$-\omega$
	1	1	-1	1	-1	1	-1

Hence, from (5) we obtain

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \tag{6}$$

Example: $r = 8$. We have $1_8 = 1, 2_8 = 4, 3_8 = 7, 4_8 = 3, 5_8 = 6, 6_8 = 2, 7_8 = 5$, and thus

$$\begin{aligned} y_1^{(8)} &= x_1 - x_4 = y_1 + y_2 + y_3 \\ y_2^{(8)} &= x_4 - x_7 = y_4 + y_5 + y_6 \\ y_3^{(8)} &= x_7 - x_3 = -(y_3 + y_4 + y_5 + y_6) \\ y_4^{(8)} &= x_3 - x_6 = y_3 + y_4 + y_5 \\ y_5^{(8)} &= x_6 - x_2 = -(y_2 + y_3 + y_4 + y_5) \\ y_6^{(8)} &= x_2 - x_5 = y_2 + y_3 + y_4 \end{aligned} \tag{7}$$

Hence, from (5) we obtain

$$M_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

It is easily verified that the matrices (6) and (8) satisfy the relations implied by (2),

$$(M_2)^7 = \mathbb{1} = (M_8)^6, \qquad M_8 M_2 = (M_2)^3 M_8 \tag{9}$$

and all 42 matrices in the faithful representation are given explicitly by (3), (6), and (8). Alternatively, of course, the computational algorithm can be used to obtain any of the 42 matrices directly via (5).

References

Littlewood, D. E. (1940). *Theory of Group Characters* (Oxford University Press, London), p. 276.

Rosen, G. (1976). *Physical Review D*, **13**, 830.